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Behaviour Modelling of Pseudo-Elastic-Plastic Material at Non-Stationary Loading

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The article is devoted to studying the behaviour of pseudo-elastic-plastic materials under significant deformations. The study of the behaviour of bodies from pseudo-elastic-plastic materials requires the development of special algorithms for calculating the stress-strain state. When constructing physical relations, it is assumed that the deformation at a point is represented as the sum of the elastic component, the jump of deformation during a phase transition, plastic deformation and deformation caused by temperature changes. A numerical method of increased accuracy based on the use of two-dimensional spline functions for solving multidimensional non-stationary problems of the theory of thermo-elasticity for bodies made of pseudo-elastic plastic materials at large deformations is proposed. A phenomenological model is constructed to describe the properties of thermo-elasticity in a point with consideration of heat generated during phase transition in geometrically nonlinear formulation. Basic equations describing the behaviour of pseudo-elastic plastic materials at significant deformations and consisting of the equation of thermal conductivity, motion, physical and geometric relations are written. Numerical examples are considered.

Key words: mathematical modelling, pseudo-elastic-plastic materials, two-

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dimensional spline functions, phenomenological model, geometric non-linearity.

Робота присвячена дослідженню поведінки псевдо-пружно-пластичних матеріалів при значних деформаціях. Вивчення поведінки тіл з псевдо-пружно-пластичних матеріалів потребує розробки спеціальних алгоритмів обчислення напружено-деформованого стану. При побудові фізичних співвідношень передбачається, що деформація в точці представлена як сума пружної складової, стрибка деформації при фазовому переході, пластичної деформації та деформації, викликаної змінами температури. Запропоновано чисельний метод підвищеної точності, заснований на використанні двовимірних сплайн-функцій для розв'язання багатовимірних нестаціонарних задач теорії термо-пружно-пластичності для тіл, виготовлених з псевдо-пружно-пластичних матеріалів, при великих деформаціях. Побудована феноменологічна модель для опису властивостей матеріалів з термо-псевдо-пружно-пластичністю в точці з урахуванням тепла, що виділяється в процесі фазового переходу, в геометрично нелінійній постановці. Записано основні рівняння, які описують поведінку псевдо-пружно-пластичних матеріалів при значних деформаціях та складаються з рівнянь теплопровідності, руху, фізичних та геометричних співвідношень. Розглянуто числові приклади.

Ключові слова: математичне моделювання, псевдо-пружно-пластичні матеріали, двовимірні сплайн-функції, феноменологічна модель, геометрична нелінійність.

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1. INTRODUCTION

Modern parts and elements of engineering and other structures and devices are made of materials that have the property of shape memory and behave pseudo-elastic-plastic. Complex deformation processes can also be caused by their uneven heating at the connected to the power factors. To model the behaviour of such structural elements, it is necessary to determine the non-stationary thermomechanical state not only of the pseudo-elastic stage of deformation, but also beyond the elastic limit. Existing numerical methods for solving such non-stationary problems usually lead to significant computational difficulties and are not always effective. Therefore, the development of methods for solving non-stationary problems of thermomechanics for spatial bodies with shape memory and pseudo-elastic plasticity is an urgent task.

Pseudo-elastic plasticity is the ability of a material under active load to accumulate deformations of a certain value at a higher temperature, and then after unloading (through a hysteresis loop) to return to its original state. The main mechanism is reverse martensitic transformation between solid phases, which can occur at room temperature. This transformation can be caused by a change in temperature or by force factors. The material is also characterized by non-linear mechanical behaviour, and large deformations. Alloys that demonstrate shape memory and pseudo-elastic plasticity are: NiTiAuCd, CuAlNi, CuSn, CuZn, NiFeGa, NiTiNb, NiNiGa, NiFeGa, NiPi, NiPeGa, NiPiGi [1, 3]. These characteristics make ASM suitable for use in various devices or as components in some advanced composite materials. NiTi alloy is the leader in most of these applications due to its structural properties.

The first alloys shape memory (ASM) were developed in the middle of the last century; however, there are no strict and reliable constant-level defining models required for engineering applications of these materials. The relationship between microscopic and macroscopic behaviour is very complex and has not yet been developed to the extent that such models and practical tasks require. This is partly due to the fairly strong dependence of mechanical reactions on temperature, load speed, strain range, geometry of the body under study, thermomechanical history and nature of the environment, as well as the interaction between these parameters.

Based on the results of the analysis, it is established that there are currently a number of models for describing the thermomechanical behaviour of alloys shape memory, pseudo-elasticity and pseudo-elastic plasticity. Most of them are based on classical representations, that is, they aim to directly describe experimental data obtained on different samples under simple and complex loads. However, as found in experimental studies, the behaviour of the material at the point of the body generally differs from the behaviour of the sample as a whole, and significant deformations may occur.

2. EXPERIMENTAL/THEORETICAL DETAILS

Statement of the problem of thermo-elastic plasticity. The main task of the non-stationary theory of thermo-plasticity is to determine the rates of displacement and components of stress and strain tensors that occur in a three-dimensional body during its loading and heating, when some elements of the body work beyond the elasticity of the material. The loading process will be considered as one that changes over time, which can cause the movement of individual parts of the body.

At first, an isotropic and homogeneous three-dimensional body V, bounded by the surface S, at the initial moment of time $t = 0$ is in a natural non-stressed state at the temperature $T_o(\theta_i)$, $i = 1, 2, 3$. Then the body is subjected to heat and load by external forces. These can be volumetric forces that affect each element of the body. Surface forces acting on one part of the body's surface.

On the second part of the body's surface, which can be fixed in a certain way, the speed of movement is set as a function of coordinates and time. Let's assume that the heating and loading of the body occur in such a way that there are deformations that can significantly affect the temperature change of this element. We will consider such loading processes and temperature levels at which the rheological properties of the material are not detected. The configuration of a body is given by the equation of the surface bounding it. In addition, you need to set the thermophysical and mechanical characteristics of the body material and the conditions for its heat exchange with the environment.

The thermophysical properties of the material are characterized by thermal and thermal conductivity coefficients, which may depend on the temperature. The heat exchange conditions are set in the form of corresponding boundary conditions, and the mechanical characteristics of the material in the study of deformation processes along rectilinear paths and the trajectory of small curvature are set in the form of instantaneous stretching diagrams of samples obtained at various fixed temperatures. In addition, the values of Poisson's coefficients ν and linear thermal expansion are set.

Based on these data, it is necessary to determine the temperature, three components of the displacement velocity vector, six components of the stress tensor and six components of the strain tensor. So we need to determine sixteen unknown functions of time and three coordinates. To do this, you need to use the equations of motion, geometric and physical equations, as well as the equation of thermal conductivity. The temperature field at an arbitrary point of the body in the presence of heat sources in it and in the case of accounting for the heat that is released during its deformation is determined by solving the thermal conductivity equation under certain initial and boundary conditions [4, 5].

After determining the temperature field for various moments of time, the components of the velocity vector of displacement and the components of stress and strain tensors that satisfy three differential equations of motion, six geometric equations, and six physical equations are searched for. These equations are solved under certain initial and boundary conditions. Initial conditions are set for all unknowns at the initial time. On the part of the surface of the body where the given forces $b(x_i, t)$, the components of the stress tensor must satisfy three boundary conditions:

$$\sigma_{in}(\alpha_k, t) = \sigma_{ij} n_j, \ i = 1, 2, 3,$$

where n_j is the guiding cosines of the external normal to the surface of the body at the corresponding point. On the rest of the surface, where the components of the displacement velocity vector are set, the displacement velocities must take the specified values:

$$v_i = V_i(x_j, t).$$

A different formulation of boundary conditions is possible, when three conditions are set on the surface of the body, taken in a certain way from the above conditions.

We will find the definition of unknowns as follows. Three components of the displacement velocity vector and six components of the stress tensor are taken as the main unknowns, for which boundary conditions are directly formulated. In this case, all components of the strain tensor are excluded from the six physical equations using geometrically nonlinear Cauchy relations, which are then determined based on the already known components of the displacement velocity vector.

When solving the non-stationary problem of thermo-plasticity, we will use the defining equations that describe non-isothermal load processes both along rectilinear trajectories and along the trajectories of small curvature deformation. After completing the task on the geometry of the deformation trajectory, you can conclude that the determining relationships used are reliable.

Theoretical Method. One of the aspects of numerical solution of General non-stationary problems for inelastic bodies is the choice of physical relations between stress and strain. This choice is consistent with experiments and is closely related to the deformation processes occurring in the body material. In General, the strain values are functions of the stress process and temperature differences, which are determined by the characteristics of the entire previous process of changes in physical factors, and not just the current values. Detailed information on this issue can be found in [7].

When constructing physical relations, it was assumed that the deformation at a point is represented as the sum of the elastic component, the jump in the deformation during the phase transition, plastic deformation and deformation caused by temperature differences.

The equations of motion of an infinitesimal volume element of a continuous medium that is deformed in the orthogonal coordinate system α_1, α_2, α_3 in are represented as:

$$\frac{\partial v_i}{\partial t} = \frac{1}{\rho}\frac{\partial \sigma_{ij}}{\partial \alpha^j} + B_i(\sigma_{mn}), \tag{1}$$

where $i, j, n, m = 1, 2, 3$, and ρ—density. Magnitude $B_i(\sigma_{mn})$, that are in the right side of equation (1) given in [5], and

$$v_i = \frac{\partial u_i}{\partial t},\ i = 1, 2, 3.$$

In general, the System of orthogonal coordinate of the strain tensor and the components of the displacement vector are connected by such nonlinear relations [6]:

$$\varepsilon_{11} = e_{11} + \frac{1}{2}\left[e_{11}^2 + \left(\frac{e_{12}}{2} + \omega_3\right)^2 + \left(\frac{e_{13}}{2} - \omega_2\right)^2\right],$$
$$\varepsilon_{12} = e_{12} + e_{11}\left(\frac{e_{12}}{2} - \omega_3\right) + e_{22}\left(\frac{e_{12}}{2} + \omega_3\right) + \left(\frac{e_{13}}{2} - \omega_2\right)\left(\frac{e_{23}}{2} + \omega_1\right). \quad (2)$$

Other components of the strain tensor are obtained from (1) by cyclic index permutation. In the case of an orthogonal rectangular coordinate system:

$$e_{11} = \frac{\partial u_1}{\partial \alpha_1}, \quad e_{22} = \frac{\partial u_2}{\partial \alpha_2}, \quad e_{33} = \frac{\partial u_3}{\partial \alpha_3},$$
$$e_{12} = \frac{\partial u_2}{\partial \alpha_1} + \frac{\partial u_1}{\partial \alpha_2}, \quad e_{23} = \frac{\partial u_3}{\partial \alpha_2} + \frac{\partial u_2}{\partial \alpha_3}, \quad e_{31} = \frac{\partial u_1}{\partial \alpha_3} + \frac{\partial u_3}{\partial \alpha_1}, \quad (3)$$
$$2\omega_1 = \frac{\partial u_3}{\partial \alpha_2} - \frac{\partial u_2}{\partial \alpha_3}, \quad 2\omega_2 = \frac{\partial u_1}{\partial \alpha_3} - \frac{\partial u_3}{\partial \alpha_1}, \quad 2\omega_3 = \frac{\partial u_2}{\partial \alpha_1} - \frac{\partial u_1}{\partial \alpha_2}.$$

With this in mind, after time differentiation in the geometrically nonlinear case, it is possible to record the following for strain rates:

$$\frac{\partial \varepsilon_{11}}{\partial t} = (1 + e_{11})\frac{\partial v_1}{\partial \alpha_1} + \left(\frac{e_{12}}{2} + \omega_3\right)\frac{\partial v_2}{\partial \alpha_1} + \left(\frac{e_{13}}{2} - \omega_2\right)\frac{\partial v_3}{\partial \alpha_1},$$
$$\frac{\partial \varepsilon_{22}}{\partial t} = \left(\frac{e_{21}}{2} - \omega_3\right)\frac{\partial v_1}{\partial \alpha_2} + (1 + e_{22})\frac{\partial v_2}{\partial \alpha_2} + \left(\frac{e_{23}}{2} + \omega_1\right)\frac{\partial v_3}{\partial \alpha_2},$$
$$\frac{\partial \varepsilon_{33}}{\partial t} = \left(\frac{e_{31}}{2} + \omega_2\right)\frac{\partial v_1}{\partial \alpha_3} + \left(\frac{e_{32}}{2} - \omega_1\right)\frac{\partial v_2}{\partial \alpha_3} + (1 + e_{33})\frac{\partial v_3}{\partial \alpha_3},$$
$$\frac{\partial \varepsilon_{12}}{\partial t} = \left(1 + \frac{e_{11}}{2} + \frac{e_{22}}{2}\right)\left(\frac{\partial v_2}{\partial \alpha_1} + \frac{\partial v_1}{\partial \alpha_2}\right) + \left(\frac{e_{12}}{2} - \omega_3\right)\frac{\partial v_1}{\partial \alpha_1} + \left(\frac{e_{12}}{2} + \omega_3\right)\frac{\partial v_2}{\partial \alpha_2} +$$
$$+ \frac{(e_{22} - e_{11})}{2}\left(\frac{\partial v_2}{\partial \alpha_1} - \frac{\partial v_1}{\partial \alpha_2}\right) + \left(\frac{e_{23}}{2} + \omega_1\right)\frac{\partial v_3}{\partial \alpha_1} + \left(\frac{e_{13}}{2} - \omega_2\right)\frac{\partial v_3}{\partial \alpha_2}, \quad (4)$$
$$\frac{\partial \varepsilon_{23}}{\partial t} = \left(1 + \frac{e_{22}}{2} + \frac{e_{33}}{2}\right)\left(\frac{\partial v_3}{\partial \alpha_2} + \frac{\partial v_2}{\partial \alpha_3}\right) + \left(\frac{e_{23}}{2} - \omega_1\right)\frac{\partial v_2}{\partial \alpha_2} + \left(\frac{e_{23}}{2} + \omega_1\right)\frac{\partial v_3}{\partial \alpha_3} +$$
$$+ \frac{(e_{33} - e_{22})}{2}\left(\frac{\partial v_3}{\partial \alpha_2} - \frac{\partial v_2}{\partial \alpha_3}\right) + \left(\frac{e_{31}}{2} + \omega_2\right)\frac{\partial v_1}{\partial \alpha_2} + \left(\frac{e_{21}}{2} - \omega_3\right)\frac{\partial v_1}{\partial \alpha_3},$$

$$\frac{\partial \varepsilon_{31}}{\partial t} = \left(1 + \frac{e_{33}}{2} + \frac{e_{11}}{2}\right)\left(\frac{\partial v_1}{\partial \alpha_3} + \frac{\partial v_3}{\partial \alpha_1}\right) + \left(\frac{e_{31}}{2} - \omega_2\right)\frac{\partial v_3}{\partial \alpha_3} + \left(\frac{e_{31}}{2} + \omega_2\right)\frac{\partial v_1}{\partial \alpha_1} +$$

$$+ \frac{(e_{11} - e_{33})}{2}\left(\frac{\partial v_1}{\partial \alpha_3} - \frac{\partial v_3}{\partial \alpha_1}\right) + \left(\frac{e_{12}}{2} + \omega_3\right)\frac{\partial v_2}{\partial \alpha_3} + \left(\frac{e_{32}}{2} - \omega_1\right)\frac{\partial v_2}{\partial \alpha_1}.$$

The system of equations (1), (4) is closed by physical relations that connect stresses and deformations.

One of the aspects of the General problem of solving non-stationary problems for NON-elastic bodies is the choice of determining relations between stresses and deformations. This choice is justified in agreement with the experiment and is closely related to the studied deformation processes. In General, the strain values are functions of the stress process and temperature differences, which are determined by the characteristics of the entire previous process of changes in physical factors, and not just the current values. Details on this issue can be found in the works of Yu. N. Shevchenko and his colleagues [4, 7].

Consider simple or close to simple deformation processes and processes of deformation along paths of small curvature. Deformation trajectories close to straight lines are those trajectories that deviate from straight lines passing through the origin of coordinates and a point on the trajectory corresponding to the beginning of the yield point, no more than a trace of delay in the vector properties of the material (5–15 field points for deformations). In this case, the smallest radius of curvature of the deformation path is greater than the delay trace. If the deviation from the straight line is greater than the delay trace, and the radius of curvature of the deformation path is less than it, then the deformation occurs along the trajectory of small curvature. In this case, the stress vector is directed tangentially to the trajectory of irreversible deformations.

Let's write down the physical relations suitable for investigation of both processes. For this purpose let's break down the process of body loading by time into separate rather small stages on each of them by means of the postulate of isotropy and the law of elastic change of volume records the relationship between stresses and deformations of the species [5]:

$$\frac{\partial \sigma_{ij}}{\partial t} = a_{ijkl}\frac{\partial \varepsilon_{kl}}{\partial t} + b_{ij}, \tag{5}$$

where

$$a_{ijkl} = 2G^*\delta_{kl}\delta_{ij} + \lambda^*\delta_{kl} - \left(1 - \frac{\delta_{kl}}{3}\right)\left(G^* - G_t\right)\frac{l_{ij}l_{kl}}{\Gamma^2},$$

$$b_{ij} = b_T \delta_{ij},\ b_T = -\left[K\alpha_t + (T - T_0)\frac{\partial}{\partial t}(K\alpha_t) \right]\frac{\partial T}{\partial t}.$$

A similar view to (4) is given of the ratios of the theory of low curvature processes. Here

$$a_{kkkk} = 2G\frac{1-\nu}{1-2\nu}, a_{kknn} = 2G\frac{\nu}{1-2\nu}, a_{knkn} = 2G,\ a_{knrr} = a_{knrs} = a_{kkrs} = 0,$$
$$b_{ij} = b_T\delta_{ij} - \sqrt{2}G\frac{S_{ij}}{S}\left(\dot{e}_{ij}^{(n)}\dot{e}_{ij}^{(n)}\right)^{1/2},\ \ k \neq n,\ r \neq s. \quad (6)$$

For further presentation of the material, we present these equations of the system in a slightly different form. We perform the elimination of strain rates in the defining relations (5) using geometrically nonlinear relations (4). Then you can write:

$$\frac{\partial \sigma_{ij}}{\partial t} = C_{ij11}\frac{\partial v_1}{\partial \alpha_1} + C_{ij21}\frac{\partial v_2}{\partial \alpha_1} + C_{ij31}\frac{\partial v_3}{\partial \alpha_1} + C_{ij12}\frac{\partial v_1}{\partial \alpha_2} + C_{ij22}\frac{\partial v_2}{\partial \alpha_2} + C_{ij32}\frac{\partial v_3}{\partial \alpha_2} +$$
$$+ C_{ij13}\frac{\partial v_1}{\partial \alpha_3} + C_{ij23}\frac{\partial v_2}{\partial \alpha_3} + C_{ij33}\frac{\partial v_3}{\partial \alpha_3} \equiv C_{ijkr}\frac{\partial v_k}{\partial \alpha_r}. \quad (7)$$

Additional symbols are entered here:

$$C_{ij11} = a_{ij11}(1 + e_{11}) + a_{ij12}(e_{12}/2 - \omega_3) + a_{ij13}(e_{31}/2 + \omega_2),$$
$$C_{ij21} = a_{ij11}(e_{12}/2 + \omega_3) + a_{ij12}(1 + e_{22}) + a_{ij13}(e_{32}/2 - \omega_1), \quad (8)$$
$$C_{ij31} = a_{ij11}(e_{13}/2 - \omega_2) + a_{ij12}(e_{23}/2 + \omega_1) + a_{ij13}(1 + e_{33}).$$

All other coefficients in (7) are obtained with a cyclic permutation of indices.

Thus, the resulting computational system of equations (1), (7) is necessary for solving non-stationary three-dimensional problems of the theory of thermo-elastic plasticity. Note that after determining the displacement rates and the components of the stress tensor, in the proposed formulation of the problem are the main unknowns, the displacements themselves are determined (by directly integrating the corresponding speeds) and the deformations using the formulas (4).

When numerically solving problems of temperature plasticity of materials, an instantaneous thermomechanical surface is set to specify the coefficients of the determining physical relations (5). It is constructed using experimental data from the study of tensile samples at various fixed temperatures. This function for some classes of pre-isotropic materials with a high degree of accuracy does not depend on the type of stress state.

Figure 1 shows a typical graph of dependences σ on ε, which are determined as a result of experiments for a pseudo-elastic-plastic material [6].

The diagrams have an initial linear section OA. The deformation processes on it are reversed. The increase and decrease in stress are in a straight line, and the deformations are small. At point *C* the elastic component of the overall deformation of the sample disappears. The deformation jump caused by the phase transition also disappears and remains only plastic deformation.

The Model is Phenomenological. Currently, a number of models are known to describe the thermomechanical behaviour of alloys shape memory (ASM). Most of them are based on classical concepts, i.e. they aim to directly describe experimental data obtained on various macro samples under spacious and complex loads. However, as established in experimental studies, the behaviour of the material at the point of the body in general differs from the behaviour of the sample as a whole. We formulate the properties of a phenomenological model that is used to describe the behaviour of bodies with pseudo-elastic-plastic materials. Deformation in a point is represented as the sum of the elastic component; deformation jump at phase transition; plastic deformation, which is subject to the theory of flow with kinematic and translational strengthening; deformation caused by temperature changes. It is assumed that material properties depend on temperature.

To describe the elastic deformation and phase transformation deformation, we will use the elastic material diagram consisting of three curvilinear or straight sections. This interpretation leads to an unstable stress-strain diagram and to describe the thermomechanical behaviour of samples of various shapes it is necessary to have a solution to the boundary problem taking into account the development of the phase transformation strain front. Not only the ambient temperature, but also the heat generated during the phase transition will be taken into account. This interpretation made it possible to propose the model from a

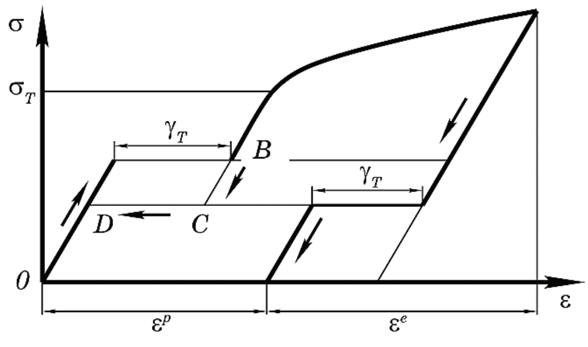

Fig. 1. Diagram of pseudo-elastic-plastic material.

single position and to describe a number of experimental data on different samples under different loading conditions, including cyclic temperature and force effects. Specific dependencies for mechanical characteristics have been established. It is shown that the phase boundary moves at a constant speed for the selected temperature. It has been established that classical material diagrams represent a curve, bending the family of material diagrams at a point, which is constructed for certain laws of change of the deformation gap front velocity [2].

The results of calculations are shown below. Figure 2 shows the typical dependence of the propagation speed of a phase transition on time. Its schedule has three sections. On the first section, the speed is zero, and on the third reaches a constant value. Between them is a section with variable speed. As a result of calculating the tangent module at each step of integration in time, there are also three characteristic sections for the integral diagram of the material.

The first section corresponds to the elastic behaviour of the material. The third characterizes the strengthening of the material. Between them is a section that resembles the behaviour of a perfectly plastic material. Similar plots have taken place during the unloading, but at a certain temperature.

Experimental Justification of the Model. This section is devoted to the experimental substantiation of a variant of the proposed model of behaviour of a material with a shape memory and thermo-pseudo-elastic-plasticity. The model makes it possible to quantify the complex interaction between the stress, temperature, strain, loading speed of the sample and the heat released during the passage of the phase transformation front passage along the sample.

Thus, the mechanical characteristics of the material in the study of deformation processes are set in the form of instantaneous diagrams of stretching samples obtained at different values of temperature, and if necessary, allow you to build an integral diagram of the material.

Processing of experimental data from [3] allowed us to construct tables and diagrams for different temperature values (respectively 100,

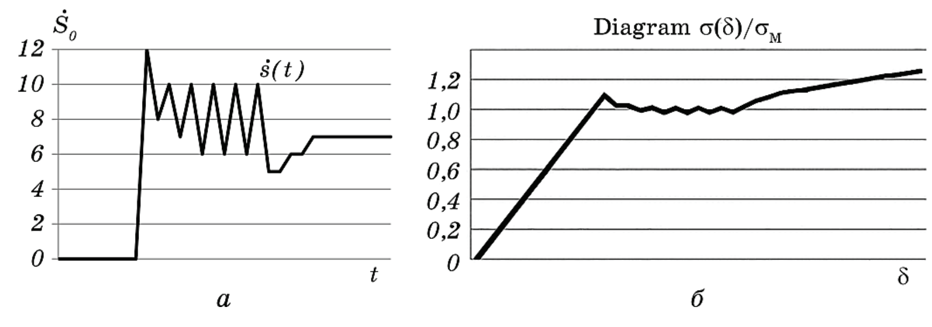

Fig. 2. Diagram of pseudo-elastic-plastic material.

90, 80, 70, 60, 50, 40, 30, 20, 10, 0°C). The results of comparison with experimental data are partially shown below in the form of diagrams. The upper lines (σ> ...°C) correspond to the active load of the sample, and the lower lines (σ<...°C) correspond to the discharge at a certain temperature.

Note that the diagrams shown in Fig. 3 are constructed for fixed temperature values without taking into account the heat that is released during the phase transformation at the material point.

The refined phenomenological model of the alloy behaviour at the material point is based on the results. Graphically, it differs from the previous model by the presence of a flow tooth on the edge of the elastic section under active load and a smooth transition of the BC section to the CD during unloading. The corresponding qualitative results are shown in Fig. 4.

Note that the diagram for the refined phenomenological model (Fig. 4, *b*) it is constructed for the same initial temperature, and the diagram shown in Fig. 4, *a*. The value of the yield tooth is modelled separately and compared with experimental data. At the same time, it is necessary to solve the thermal conductivity equations separately and determine additional strain and stress values based on temperature differences [2]. In fact, the flow tooth diagram is a projection of a trajectory that follows an instantaneous thermo-mechanical surface.

To obtain the calculation formulas of the refined model, an auxiliary problem related to the development of an instantaneous thermo-

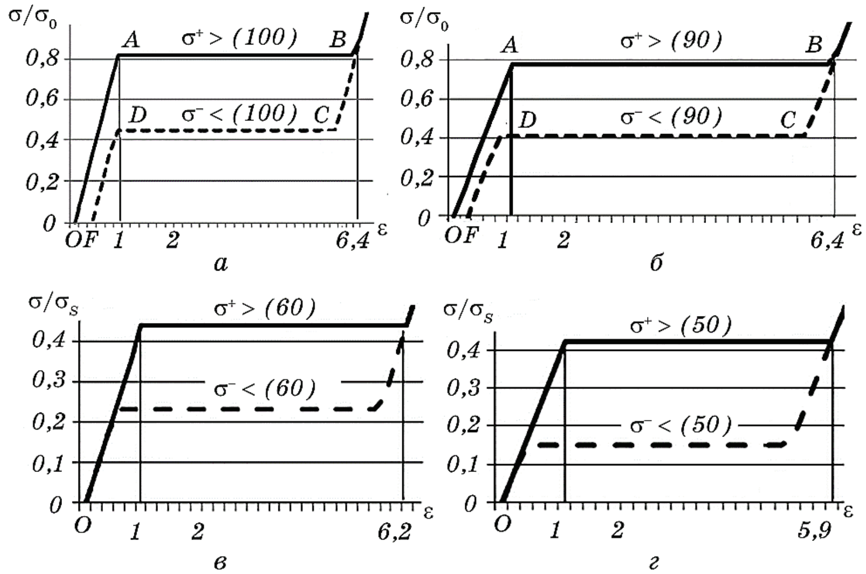

Fig. 3. Local material diagrams at 100°C (*a*), 90°C (*b*), 60°C (*c*), 50°C (*d*).

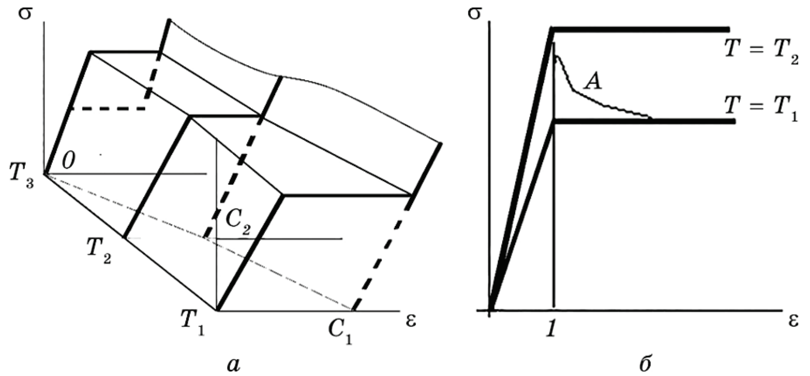

Fig. 4. Local material diagram (a, b).

mechanical surface was considered. Let the coordinates ε, T, σ of four points be given in three-dimensional space $P_i(\varepsilon_i, T_i, \sigma_i)$, $i = 1, 2, 3, 4$.

The equation of the thermomechanical surface passing through these points is written down as follows

$$\sigma = a\varepsilon + bT + c\varepsilon T + d. \tag{9}$$

Unknown coefficients a, b, c, d are searched for from a system constructed based on this expression for the given four points P_i of the instantaneous thermo-mechanical surface. After their calculation, you can build such a surface from two adjacent diagrams obtained for $T = T_1$ and $T = T_2$, on the interval $T \in [T_1; T_2]$. Under active load, the total thermo-mechanical surface consists of three separate surfaces. This is the surface of the elastic part, the surface where deformations caused by phase transformation and the plastic part of the surface are jumping. Thermo-mechanical surface is simulated in the same way during unloading.

Thus, having an analytical expression for such surface and the law by which the temperature changes at a particular point of the sample, it is possible to refine the local diagram of the material.

Generalization of Physical Relations. We will make some generalizations of known physical relations (flow theory with kinematic and translational reinforcement, and others) in the case of thermo-pseudo-elastic-plastic materials.

Based on the results of the previous section, the total strain tensor is represented as the sum of the elastic component ε_{ij}^{e}, the jump of deformation during the phase transition ε_{ij}^{T}, plastic deformation ε_{ij}^{p}, and deformation $\varepsilon_{ij}^{\theta}$ caused by temperature changes.

As a result, you can write

$$\varepsilon_{ij} = \varepsilon_{ij}^{e} + \varepsilon_{ij}^{T} + \varepsilon_{ij}^{p} + \varepsilon_{ij}^{\theta}. \tag{10}$$

Elastic deformations are determined using Hooke's law. Deformations caused by temperature changes satisfy the law of linear thermal expansion. The jump of deformations at the phase transition will be defined as follows:

$$\varepsilon_{ij}^{T} = \varepsilon_{T}(T)\frac{\partial f_{\gamma}(\sigma_{ij})}{\partial \sigma_{ij}}, \tag{11}$$

where is the function:

$$f_{\gamma}(\sigma_{ij}) = 0 \tag{12}$$

sets the limits of a certain surface in the stress space [4, 8]. When passing through this surface, the deformation caused by the phase transition increases with a jump. We will set it as follows:

$$f_{\gamma}(\sigma_{ij}) = \sqrt{\frac{3}{2}(S_{ij}S_{ij})} - \sigma_{\gamma}(T), \tag{13}$$

where

$$S_{ij} = \sigma_{ij} - \delta_{ij}\sigma, \ \ \sigma_{\gamma}(T) = \begin{cases} \sigma_{M}(T), \ S_{ij}\dot{S}_{ij} > 0, \\ \sigma_{m}(T), \ S_{ij}\dot{S}_{ij} < 0. \end{cases} \tag{14}$$

From here we finally get

$$\varepsilon_{ij}^{T} = \frac{3\varepsilon_{T}(T)}{2\sigma_{\gamma}(T)}S_{ij}. \tag{15}$$

Plastic deformations must satisfy the relation of one or another theory of plasticity. We present the defining physical relations of some theories of plasticity. Figure 1 also shows the elastic discharge of the sample along a straight *BC*, which is assumed to be parallel to the *OA* line. This representation of the unloading mechanism shows in General terms only the actual process of material deformation with small deformations. For large deformations (10% or more), the deformation process of the samples will be significantly nonlinear.

Note that in General, the defining relations for thermo-pseudo-elastic-plastic materials can also be represented as formulas (5).

Method of Splitting by Geometric Properties. The solution of the main calculation system (1), (7), recorded in the previous paragraph, will be performed using the method of component-by-component splitting, which allows you to build three-dimensional non-stationary elastic plasticity problems to a sequentially solved system of three one-dimensional problems at each step in time.

Let's introduce the time ω_r grid taking into account the small steps:

$$\omega_r = \left\{ \begin{array}{c} t_p;\ t_{p+1/3} = t_p + \tau_1;\ t_{p+2/3} = t_{p+2/2} + \tau_2; \\ t_{p+1} = t_{p+2/3} + \tau_3;\ \tau = \tau_1 + \tau_2 + \tau_3; \\ t_0 = 0;\ p = 0, 1, 2, \ldots \end{array} \right\}, \quad (16)$$

$$\frac{\partial \mathbf{W}}{\partial t} = A_k \frac{\partial \mathbf{W}}{\partial \alpha^k} + \gamma_k \mathbf{B}, \quad t \in [t_{p+(k-1)/3}, t_{p+k/3}],\ k = 1, 2, 3. \quad (17)$$

The idea of the component splitting method is that instead of (1), (7) at the full time integration step τ ($t \in [t_p;\ t_{p+1}]$), three equivalent one-dimensional systems are solved sequentially. Each such system is solved at its own fractional step in time. At each such fractional step, the one-dimensional coordinate system of equations will have the form where $\mathbf{W}$ is the vector whose components will be the desired values

$w_1 = v_1$;	$w_4 = \sigma_{11}$;	$w_7 = \sigma_{12}$;	$w_{10} = \varepsilon_{11}$;	$w_{13} = \varepsilon_{12}$;
$w_2 = v_2$;	$w_5 = \sigma_{22}$;	$w_8 = \sigma_{13}$;	$w_{11} = \varepsilon_{22}$;	$w_{14} = \varepsilon_{13}$;
$w_3 = v_3$;	$w_6 = \sigma_{33}$;	$w_9 = \sigma_{23}$;	$w_{12} = \varepsilon_{33}$;	$w_{15} = \varepsilon_{23}$.

Equivalence of the split system (17) to the system of equations (1), (5), (7) occurs when the condition is met $\gamma_1 + \gamma_2 + \gamma_3 = 1$. The matrices A_k, $k = 1, 2, 3$ and vector $\mathbf{B}$ are defined simply by comparing (17) with (1), (7).

Let's denote $\mathbf{W}^{p+k/3}$ the vector $\mathbf{W}$ calculated for the moment of time $t = t^{p+k/3}$ ($k = 0, 1, 2, 3$). We will introduce similar notation for the vector $\mathbf{B}$. When passing from equations (17) to the difference scheme of their solution, the time derivatives are replaced by difference relations, and the coordinate derivatives are represented by a linear combination of derivatives specified at the previous time step and at the stage at which the solution is sought. As a result, the calculation scheme can be written as follows:

$$\begin{gathered} \frac{1}{\tau}\left(\mathbf{W}^{p+1/3} - \mathbf{W}^{p}\right) = \alpha\Lambda_1 \mathbf{W}^{p+1/3} + \beta\Lambda_1 \mathbf{W}^{p} + \gamma_1 \mathbf{B}^{p}, \\ \frac{1}{\tau}\left(\mathbf{W}^{p+2/3} - \mathbf{W}^{p+1/3}\right) = \alpha\Lambda_2 \mathbf{W}^{p+2/3} + \beta\Lambda_2 \mathbf{W}^{p+1/3} + \gamma_2 \mathbf{B}^{p+1/3}, \\ \frac{1}{\tau}\left(\mathbf{W}^{p+1} - \mathbf{W}^{p+2/3}\right) = \alpha\Lambda_3 \mathbf{W}^{p+1} + \beta\Lambda_3 \mathbf{W}^{p+2/3} + \gamma_3 \mathbf{B}^{p+2/3}, \end{gathered} \quad (18)$$

where $\alpha + \beta = 1$ and the differential operators $\Lambda_n \mathbf{W}$ are entered as follows:

$$\Lambda_k \mathbf{W} = \frac{A_k}{h_k} \lambda_k(\mathbf{W}), k = 1, 2, 3. \quad (19)$$

Here $\lambda_k(\cdots)$ is the difference operator, h_k—integration steps by coordinates ($k = 1, 2, 3$).

In the case where $\alpha = 0$, $\beta = 1$, the scheme (8) will be explicit. When $\alpha = 1$, $\beta = 0$, term (8) gives an implicit schema. If $\alpha = \beta = 1/2$, then there is a Crank–Nicholson's scheme, which, unlike the two previous schemes of the first order of approximation in time, has a second order of approximation.

The algorithm for solving system (8) at the full time step consists of three steps. First, an auxiliary vector $\mathbf{W}^{p+1/3}$ is found based on the first equation. In this case, the vector $\mathbf{W}^{p}$ is already known either from the previous step in time or from the initial conditions. Then a vector $\mathbf{W}^{p+2/3}$ is defined from the second equation, and then vector $\mathbf{W}^{p+1}$ is found from the third equation.

To implement the described procedure for constructing the solution of the system (18) at the full time step, it is necessary to approximate how the differential operators (19) are constructed. To do this, enter the grid by coordinates ω_h:

$$\omega_h = \left\{ \begin{array}{l} (\alpha_i^1; \alpha_j^2; \alpha_k^3); \\ \alpha_i^1 = \alpha_{i-1}^1 + h_1,\ i = 1, 2, \ldots, N_1; \\ \alpha_j^2 = \alpha_{j-1}^2 + h_2,\ j = 1, 2, \ldots, N_2; \\ \alpha_k^3 = \alpha_{k-1}^3 + h_3,\ k = 1, 2, \ldots, N_3 \end{array} \right\}, \tag{20}$$

$$h = \max(h_1, h_2, h_3).$$

Interpolation of the decoupling W_m, $m = 1, 2, \ldots, 15$, between the nodes of the grid ω_h and approximation of the difference operators $\lambda_n(\ldots)$ in the nodes of this grid will be carried out both using cubic B-splines and on the basis of stressed splines. In the future, in order to obtain compact expressions for the desired functions $W_m(a^1, a^2, a^3, t^{p+n/3})$ in brackets, we will only give the variable that is being integrated at this fractional step in time n, that is $W_m^{p+n/3}(\alpha^n)$, either $W_m(\alpha^n)$. Other coordinates are considered to be fixed.

3. RESULTS AND DISCUSSION

Numerical Results. Consider the first problem of propagation of a slow phase transition wave in a rod when it is stretched.

Determine the speed of phase transition boundary propagation along the rod $x \in [0; L]$. At edge $x = 0$, the speed $v = V_0$ is set at which the edge of the specimen is stretched. The other edge $x = L$ is fixed here $v = 0$.

The main values are: displacement speed along the axis of the rod $v(x, t)$, stress $\sigma(x, t)$, deformation $\varepsilon(x, t)$, and temperature $T(x, t)$.

To determine the unknown quantities write down the system:

$$\rho \frac{\partial v}{\partial t} = \frac{\partial \sigma}{\partial x}, \ \frac{\partial \varepsilon}{\partial t} = \frac{\partial v}{\partial x}(1 + \varepsilon),$$

$$\sigma = \begin{cases} E_1\varepsilon - K\alpha_T(T - T_0), & \varepsilon \in [0, \varepsilon_S], \\ E_2(\varepsilon - \varepsilon_S) + \sigma_S - K\alpha_T(T - T_0), & \varepsilon \in [\varepsilon_S, \varepsilon_C], \\ E_3(\varepsilon - \varepsilon_c) + \sigma_C - K\alpha_T(T - T_0), & \varepsilon \in (\varepsilon_C, \infty), \end{cases} \quad (21)$$

$$\frac{\partial T}{\partial t} = a^2 \frac{\partial^2 T}{\partial x^2} + W_*.$$

Here ρ is the density of the material, the modules E_1, E_2, ..., E_3 of the local diagram of the material, as well as the coefficients of linear thermal expansion and thermal conductivity α_T, a. The function that takes into account W_* the heat generated in the body during the phase transition (in the diagram of the material from the position A to C or from A to B) is indicated by.

Let's go to the non-zero normalized values of the desired values, for which we will save the previous designations. Through denotes v_*, T_*, x_*, t_* some set values for the displacement rate, temperature, space coordinates and time, σ_{ST}, $\varepsilon_{ST}\left(\sigma_{ST} = E_1(T_*)\varepsilon_{ST}\right)$—the yield strength of the material in terms of heat and strain, which are important at temperature $T = T^*$.

Then an explicit difference system equivalent to a complete system of equations can be written as follows:

$$v^{p+1} = v^p + \tau k_{1*}\lambda(\sigma^p), \varepsilon^{p+1} = \varepsilon^p + \tau\lambda(v^p)(1 + k_{4*}\varepsilon^p),$$
$$T^{p+1} = T^p + \tau k_{3*}\mu(T^p) + \tau W_*{}^p t_*. \quad (22)$$

Note that the voltage value at any given time can be determined directly by the corresponding formula with (21).

In the calculation formulas, a designation has been introduced for pick operators who approximate the first and second coordinate derivatives. For example, the simplest operators for central nodes have the following form

$$\lambda(y_i^p) = \frac{y_{i+1}^p - y_{i-1}^p}{2h}, \ \mu(y_i^p) = \frac{y_{i+1}^p - 2y_i^p + y_{i-1}^p}{h^2} \quad (23)$$

and give the second order of accuracy on the grid:

$$\omega_h \left\{x_i; \ i = 1, 2, \ldots, n-1\right\}.$$

On the edges of the rod, one-way difference operators are formed. Accordingly, for the first derivatives (first order of accuracy):

$$x = 0,\ \lambda(y_0^p) = \frac{y_1^p - y_0^p}{h},\ x = L,\ \lambda(y_n^p) = \frac{y_n^p - y_{n-1}^p}{h} \tag{24}$$

and for second derivatives (second order of accuracy):

$$\begin{aligned} x = 0,\quad & \mu(y_0^p) = \left(2y_0^p - 5y_1^p + 4y_2^p - y_3^p\right)/h^2, \\ x = L,\ & \mu(y_n^p) = \left(2y_n^p - 5y_{n-1}^p + 4y_{n-2}^p - y_{n-3}^p\right)/h^2. \end{aligned} \tag{25}$$

A high accuracy of derivative calculations is reached by the addition of difference formulas obtained into practice [5].

The following expressions were obtained for approximation of the first derivatives

$$\lambda(y_i^p) = \frac{n_0\left[y_{i+1}^p - y_{i-1}^p\right] - k_0\left[y_{i+2}^p - y_{i-2}^p\right]}{12h},\ i = 2, 3, \ldots, n-2,$$

$$\begin{aligned} \lambda(y_0^p) &= \frac{-k_1 y_0^p + k_2 y_1^p - k_3 y_2^p + k_4 y_3^p}{6h}, \\ \lambda(y_1^p) &= \frac{-k_4 y_0^p - k_5 y_1^p + k_6 y_2^p - k_0 y_3^p}{6h}, \\ \lambda(y_{n-1}^p) &= \frac{k_4 y_n^p + k_5 y_{n-1}^p - k_6 y_{n-2}^p + k_0 y_{n-3}^p}{6h}, \\ \lambda(y_n^p) &= \frac{k_1 y_n^p - k_2 y_{n-1}^p + k_3 y_{n-2}^p - k_4 y_{n-3}^p}{6h}. \end{aligned} \tag{26}$$

Approximation of the second derivatives at the numerical solution of the heat conductivity equation is carried out as follows:

$$\begin{aligned} \mu\left(T_i^p\right) &= -m_0\left(T_{i+1}^p - 2T_i^p + T_{i-1}^p\right)/h^2,\ i = 1, 2, \ldots, n-1, \\ \mu\left(T_0^p\right) &= \left(m_1 T_0^p + m_2 T_1^p + m_3 T_2^p + m_0 T_3^p\right)/h^2, \\ \mu\left(T_n^p\right) &= \left(m_1 T_n^p + m_2 T_{n-1}^p + m_3 T_{n-2}^p + m_0 T_{n-3}^p\right)/h^2. \end{aligned} \tag{27}$$

Formulas (26) and (27) use coefficients that have been determined using spline functions. If cubic B-splines with the fourth order of approximation are used, then

$$n_0 = 8,\ k_0 = 1,\ k_1 = 11,\ k_2 = 18,\ k_3 = 9,\ k_4 = 2,$$
$$k_5 = 3,\ k_6 = 6,\ m_0 = -1,\ m_1 = 2,\ m_2 = -5,\ m_3 = 4.$$

When using stressed splines, which have the fifth order of approximation:

$$n_0 = 7,9136,\ k_0 = 0,9568,\ k_1 = 11,2646,\ k_2 = 18,4641,$$
$$k_3 = 9,1344,\ k_4 = 1,9349,\ k_5 = 3,0870,\ k_6 = 5,9787,$$
$$m_0 = -0,9817,\ m_1 = 2,1856, m_2 = -5,3529,\ m_3 = 4,149.$$

The boundary conditions at the edges of the heat transfer equation are formulated as free heat exchange conditions:

$$\partial T/\partial x = 0,\ x \in [0; L].$$

Then from the second and fifth formulas (26), respectively, we will get the calculated formulas along the edges of the rod:

$$T_0^p = \left(k_2 T_1^p - k_3 T_2^p + k_4 T_3^p\right)/k_1,\quad T_n^p = \left(k_2 T_{n-1}^p - k_3 T_{n-2}^p + k_4 T_{n-3}^p\right)/k_1.$$

The results of the numerical solution of the problem are shown in Figures 5, *a–c*. The distribution of strain and stress along the rod for different time moments is shown here:

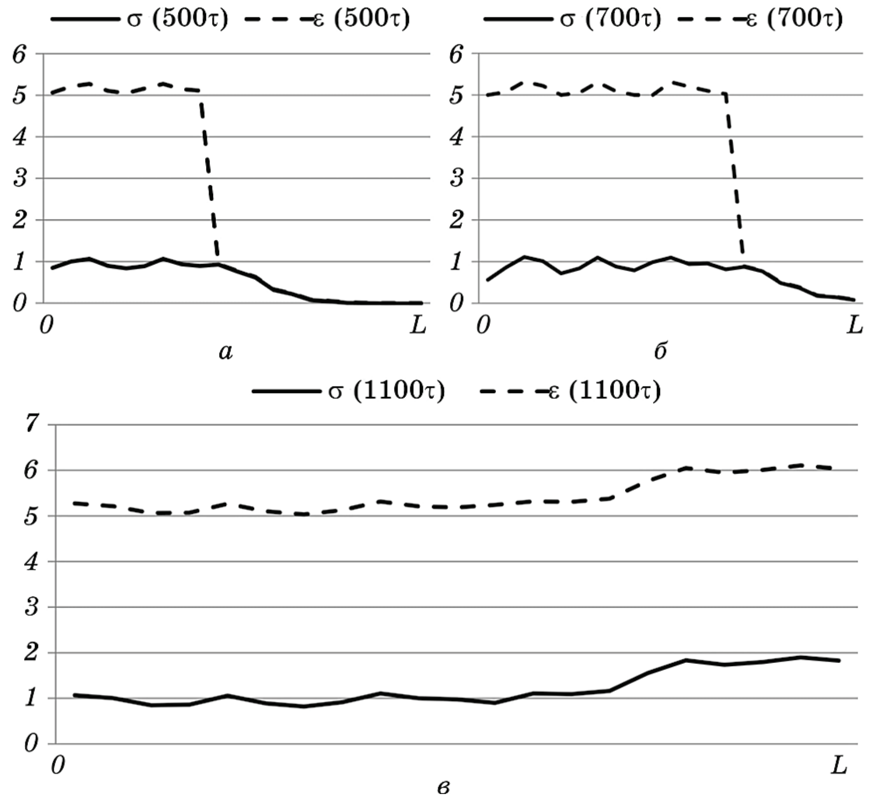

Fig. 5. Slow wave propagation (*a–c*).

Changes in time of the temperature field due to the heat generated by the phase transition sequence (jump from point A to point B in the material diagram) are shown in Fig. 6.

Let us determine the influence of shape memory of the material on rod behaviour. Let's consider the second task on loading, the following unloading of a one-dimensional rod and its further heating in which influence of memory of the form of a material in a material point on behaviour of a rod as a whole is simulated.

At the first stage, the solution of the previous problem was obtained at $\tau = 0.0025$; $T = 10°C$.

For the time interval $0 < t \le 1100\tau$ at the edge $x = 0$, the speed $\upsilon = V_0$ ($V_0 = -1$), at which the specimen is stretched, is set.

The edge $x = L$ is fixed and here the travel speed is zero. Deformation and stress distributions $t = 2400\tau$ are shown in Fig. 7, *a*. Over the entire length, the deformation is almost constant. We use these results as initial conditions in the second stage.

In Figure 7, *b* line *1* received in geometrical linear statement, and line 2 corresponds to geometrically nonlinear variant.

In the second stage, the rod is heated uniformly along its length under constant boundary conditions. The distributions of strain and stress for the corresponding time and temperature moments are shown in Fig. 8.

Results and Discussion. The above results show the behaviour of

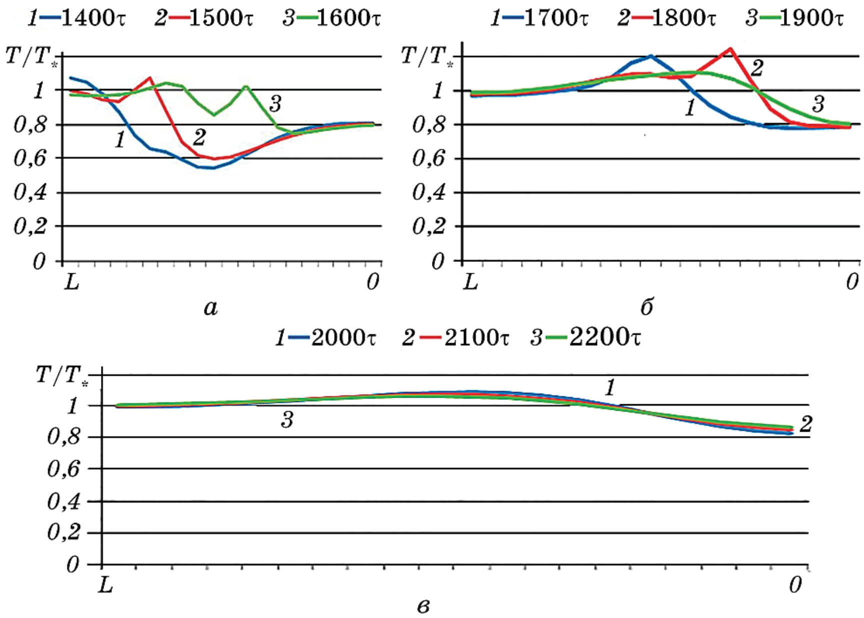

Fig. 6. Changes in the temperature field along the rod (*a*–*c*).

Ni–Ti alloy, which is characterized by form memory. As the temperature of deformation increases, the deformation caused by the phase transition decreases significantly. Almost all along the rod, they're equal to zero. Residual deformations in some points of the body are caused by the fact that plastic deformations can also occur in the material at a temperature (Fig. 8, *b*).

4. CONCLUSION

The paper proposes a new phenomenological model for describing properties with form memory at significant deformations. The model takes into account the heat that is released during phase transitions in the material points of the body. This allowed to describe a number of

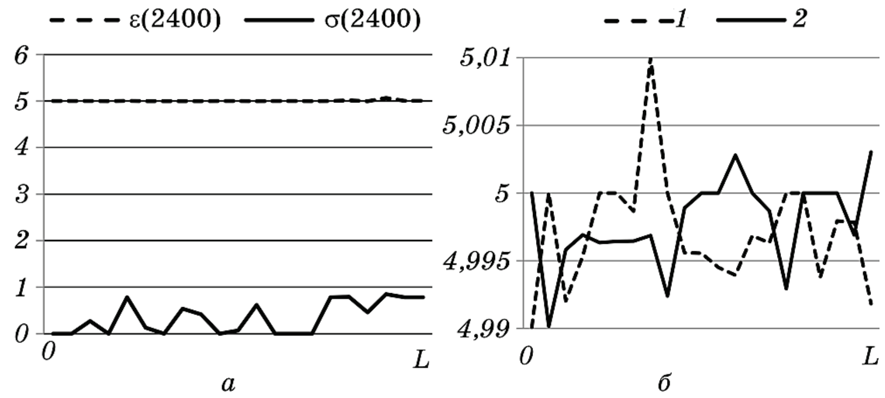

Fig. 7. Distribution of deformation (*a*) and stress before heating (*b*).

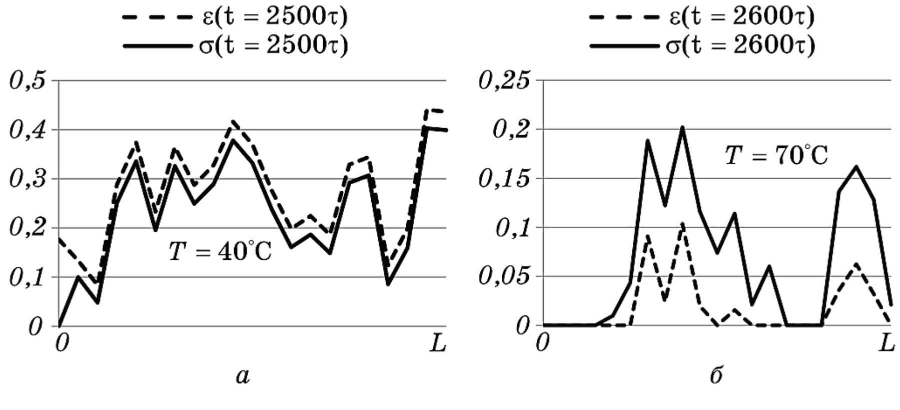

Fig. 8. Distribution by length of rod of deformation (*a*) and pressure at heating for various moments of time and temperatures (*b*).

experimental data on different samples at different temperatures and loading conditions, to obtain the necessary constants of the proposed phenomenological model.

In case of deformation of thermopseudo-elastic material the physical relations of plasticity theory (flow theory with kinematic and translational strengthening) are generalized that allowed to apply the developed phenomenological model at the solution of thermomechanics problems on continuum level.

A new variant of an effective method for solving non-stationary spatial problems of thermomechanics is developed. It is based on the use of the idea of splitting a complete system of equations by geometric properties. It is used for approximation of unknown functions and their derivatives by coordinates of two-dimensional tension splines. This approach allowed increasing the accuracy of the method's approximation to the fourth order (by two orders of magnitude). This made it possible to select a larger grid in terms of coordinates compared to the finite difference method provided that the same accuracy of calculations is achieved.

To increase to the third order of method's approximation by time an iterative procedure is offered, coincides. The initial approximation for it will be the results calculated using the explicit splitting method formulas. It is revealed that the sequence of approximate solutions of the problem obtained by reducing the crocus integration in time by half coincides with the exact one. It is shown that three consecutive approximations can be used to estimate an accurate solution.

The effectiveness of the generalized method was studied and the accuracy of the obtained results was assessed. In case of application of implicit schemes of the method of splitting by geometrical properties the similarity of the corresponding iteration procedure is established.

A new class of problems on non-stationary deformation of spatial bodies from alloys with shape memory properties, thermopseudo-submersible-plasticity, is put and solved on the basis of the proposed method.

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