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Approximation of the Yield Strength in Function of Young's Modulus and Poisson's Ratio from Continuous Needle Indentation Testing

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Based on mathematical assumptions by Love and Sneddon with co-workers, a yield strength-relationship result from the indentation of a needle of sharp conical tip in elastic materials is determined. This study offers mathematical expressions of the yield strength of solid material, when the penetrator is a sharp needle with a cone tip. According to Love (1939) and by Sneddon (1965) workers, we have deduced the relationship between the yield strength, the Young's modulus (E), and the Poisson's ratio (γ) for the tested material and the semi-included angle of the sharp tip of the needle indenter (θ). Firstly, mathematical concepts of the right circular cone are established. Secondly, the yield strength formula for the bulk material was derived. Thirdly, an approach of the yield strength resulted from the indentation of the sharp conical needle is presented. Finally, yield strength expressions resulted from the indentation of the sharp conical needle of different semi-angles are determined and collected in tables at the last.

Key words: pressure, cone contact, geometric modelling, indentation, yield

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strength.

На основі математичних припущень Лава та Снеддона із колегами визначено залежність межі плинності, одержану в результаті вдавлювання голки з гострим конічним кінчиком в еластичні матеріали. Це дослідження пропонує математичні вирази межі плинності твердого матеріалу, коли пенетратор є гострою голкою з конічним кінчиком. Згідно з Лавом (1939) та Снеддоном (1965), ми вивели залежність між межею плинності, модулем Юнга (E) та Пуассоновим коефіцієнтом (γ) випробуваного матеріалу та напіввключеним кутом гострого кінчика голчастого індентора (θ). По-перше, встановлено математичні концепції правильного кругового конуса. По-друге, одержано формулу межі плинності об'ємного матеріалу. По-третє, представлено підхід щодо визначення межі плинності, одержаної в результаті вдавлювання гострої конічної голки. Нарешті, визначено вирази межі плинності, одержані в результаті вдавлювання гострої конічної голки з різними напівкутами, що узагальнено у таблицях.

Ключові слова: тиск, конічний контакт, геометричне моделювання, вдавлювання, межа плинності.

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1. INTRODUCTION

The study of the deformation between materials in contact can reveal some information about their mechanical properties. The first try of materials contact is considered by Heinrich Hertz for two contacting spheres (Hertz, 1881–1882).

The material properties are an important condition for the investigation, design and safety assessment of engineering structures. There is a grand deal of effort has been directed at the way to develop the methods of determining of the mechanical properties of materials. Indentation tests have been applied to establish the material's strength [1] since many years ago. In addition, methods of indentation tests have been developed and are extensively used to determine some material properties, such as Young's modulus and hardness, *etc.* [2–13].

As of penetration, it is common and efficient use; a grand importance has grown in the needle's indentation while penetrating a material, to minimize the impact on the penetrated surface and improve the performance of the needle tip.

The indentation test is a large technique used for characterizing the mechanical properties of materials at the nano/microscale; it has been successfully used to evaluate the strength of surface material due to its ability to penetrate the materials on a micro- or nanoscale.

While the indenter is a needle of a cone tip, the horizontal project result from the indentation is circle form. According to Love (1939)

and by Sneddon (1965) workers, the yield strength relationship of bulk material was determined in function of the Young's modulus, E , and the Poisson's ratio, γ , for the tested material and the semi-included angle of the conical indenter, θ .

2. THEORETICAL BACKGROUND

2.1 Cartesian Equation of the Cone Form

The right circular cone is a three-dimensional geometric shape; it has a pointy end on one side and a flat circular surface on the other side. It has a line that touches the apex point of the cone in a perpendicular of the centre of its circular base. Furthermore, its apex lies just above the centre of its base.

In Cartesian co-ordinate system, a right cone of a circular base of diameter $D = 2R$ and height H , oriented along the Z -axis, with a vertex pointing down of half angle, θ and a base located at $Z = H$ (Fig. 1), can be described by the equation [12]:

$$x^2 + y^2 - (D / 2H)^2 Z^2 = 0 . \quad (1)$$

2.2 Indentation of Elastic Solids with Rigid Cones

Boussinesq provide the earliest formula of the distribution of pressure within the elastic half-space that deformed by a rigid punch [17]. Due the easy machinability of conical form tips, conical indentation methods have been generally developed. Compared with the other indentation technique, the sharp indentation using a conical form has

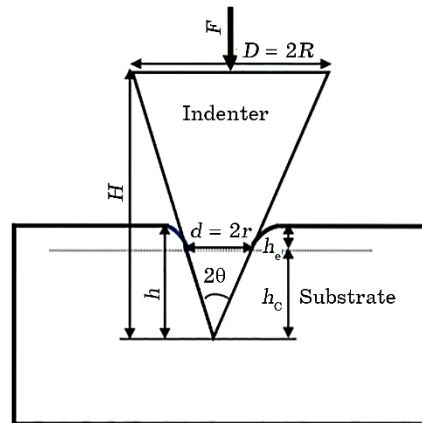


Fig. 1. Right cone of a circular base in Cartesian co-ordinate system.

a large variety of indented depth.

The static pressure on rigid materials is defined as the resistance of the permanent shape penetration on the surface of the tested material when a constant compressive load is applied [12]; it is defined by the report of the load F , applied to the indenter on the contact surface S :

$$P_c = \frac{F}{S}. \quad (2)$$

In the cone indentation method, which is adopted in this study, a right needle of a cone tip of a circular base of radius R , and height H , is pressed into the surface of the tested material (Fig. 2). The result pressure value P_c is obtained as the ratio of the applied load to the surface area of the project of the resulting imprint.

While the projected surface of the resulting imprint takes the form of a circular disk (Fig. 3) of half axes, r , [20]:

$$S = \pi r^2. \quad (3)$$

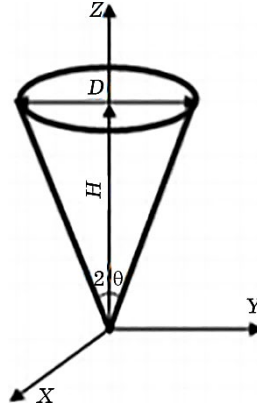


Fig. 2. Right cone of a circular base in Cartesian co-ordinate system.

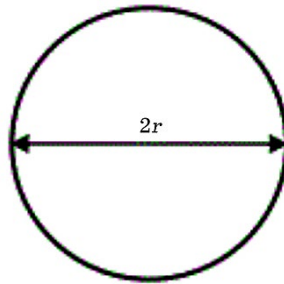


Fig. 3. Projected surface of resulting imprint.

The hardness of the cone indentation can be expressed as follows:

$$P_c = \frac{F}{\pi r^2}. \quad (4)$$

As shown in Fig. 2, the true distance of the indenter tip below the original surface of the test substrate, h , can be written in function of the contact depth, h_c , and is the deflection of the tested surface at the contact edge of the indenter, h_e , as follows:

$$h_e \approx 0. \quad (5)$$

In addition, in Figure 2, it is evident that it can be written the half angle θ in a function of the height H , the radius of the cone, radius of the imprint and base R and the contact depth, h_c as

$$\tan \theta = \frac{r}{h_c} = \frac{R}{H}. \quad (6)$$

When the deflection of the tested surface at the contact edge of the indenter is neglected, $h_e \approx 0$, the true distance of the indenter tip below the original surface of the test substrate, h , becomes:

$$h = h_c. \quad (7)$$

When replacing Eq. (7) into Eq. (6), the radius of the imprint, r , can be determined as function of the height H , the radius R , and the contact depth, h , as

$$\tan \theta = \frac{r}{h} \approx \frac{R}{H} \rightarrow r \approx h \frac{R}{H}. \quad (8)$$

Then, the static pressure, P_c , becomes according of the load F , the height H , the radius, R , and the contact depth h as follows:

$$P_c = \frac{H^2 F}{\pi (hR)^2}. \quad (9)$$

2.3 Load-Depth Relation of Elastic Indentation

The mathematical basics of the punch penetration on an elastic half-space are more than a century old.

In 1885, Boussinesq presented a resolution of contact between an elastic continuum and a solid of revolution, then, Love found a solution for the important cases of conical and cylindrical [14, 15] punches, and later, Harding and Sneddon [16] established an analytical

procedure for deriving load-displacement relations for a punch of arbitrary axisymmetric shape.

The earliest study focused on the indentation test as an elastic half-space problem is according to results by Boussinesq, Love [14–23] and Sneddon successively derived the load–depth relation of elastic indentation with a cone.

In elastic contact hypothesis, specifically, the analyses of Love and Sneddon [14–18] for contact of an isotropic elastic half-space by rigid indenters of various geometry, the elastic loading of a frictionless rigid cone on to an elastic half-space F , have been determined as follows:

$$F = \frac{2Eh^2}{\pi(1-\gamma^2)} \tan(\theta), \quad (10)$$

where E and γ are Young's modulus and Poisson's ratio, respectively, of the tested material, h —the true distance of the indenter tip below the original surface of the test substrate, and θ is the semi-included angle of the conical indenter.

By replacing Eq. (10) into Eq. (9), the yield strength gets the following expression:

$$\sigma_y = \frac{2H^2E \tan\theta}{(\pi R)^2 (1-\gamma^2)} = 0.2026 \frac{H^2E \tan\theta}{(R)^2 (1-\gamma^2)}. \quad (11)$$

Because $\tan(\theta) = \frac{r}{h} = \frac{R}{H}$, the yield strength becomes

$$\sigma_y = \frac{0.2026E}{(1-\gamma^2) \tan\theta}. \quad (12)$$

For a sharp needle of a cone form, the semi-included angle is very small, $\theta \leq \frac{\pi}{n}$ with $n \geq 18$, so, we can write

$$\tan\theta \approx \sin\theta. \quad (13)$$

Then, the relationship between the applied load F on the needle, the true distance of the indenter tip below the original surface of the test substrate, h , the Young's modulus, E , the Poisson's ratio, γ , and the semi-included angle of the conical indenter, θ , becomes as follows:

$$F = \frac{2Eh^2}{\pi(1-\gamma^2)} \sin(\theta). \quad (14)$$

TABLE 1. Variation of the applied load (F) and yield strength (σ_y) with semi-included angle of the needle indenter (θ).

Semi-included angle (θ)	10°	8°	6°	4°	2°
Applied load F	$\frac{0.11Eh^2}{(1-\gamma^2)}$	$\frac{0.88Eh^2}{(1-\gamma^2)}$	$\frac{0.66Eh^2}{(1-\gamma^2)}$	$\frac{0.44Eh^2}{(1-\gamma^2)}$	$\frac{0.22Eh^2}{(1-\gamma^2)}$
Yield strength	$\frac{1.166E}{(1-\gamma^2)}$	$\frac{1.455E}{(1-\gamma^2)}$	$\frac{1.938E}{(1-\gamma^2)}$	$\frac{2.904E}{(1-\gamma^2)}$	$\frac{5.805E}{(1-\gamma^2)}$

When replacing Eq. (13) into Eq. (12), the yield strength expression becomes as follows:

$$\sigma_y = \frac{0.2026E}{(1-\gamma^2)\sin(\theta)}. \quad (15)$$

To determine the suitable semi-angle to find the yield hardness, we take, for example, the values from Table 1.

In addition, where the semi-included angle of the needle tip is very small, we can consider that

$$\sin(\theta) \approx \pi/n \text{ with } n \geq 18. \quad (16)$$

By replacing Eq. (16) into Eq. (14), the applied load F on the needle indenter becomes as function of the Young's modulus E , the Poisson's ratio γ , the reel number n , and the resulting displacement h as

$$F = \frac{2Eh^2}{n(1-\gamma^2)}. \quad (17)$$

In addition, when substituting Eq. (16) into Eq. (15), the hardness

TABLE 2. Expressions of the applied load (F) and yield strength (σ_y) with semi-included angle of the needle indenter ($\sin(\theta) \approx \pi/n$).

Real number n	18	22.5	30	45	90
Applied load F	$\frac{0.11Eh^2}{(1-\gamma^2)}$	$\frac{0.88Eh^2}{(1-\gamma^2)}$	$\frac{0.66Eh^2}{(1-\gamma^2)}$	$\frac{0.44Eh^2}{(1-\gamma^2)}$	$\frac{0.22Eh^2}{(1-\gamma^2)}$
Yield strength σ_y	$\frac{1.161E}{(1-\gamma^2)}$	$\frac{1.141E}{(1-\gamma^2)}$	$\frac{1.935E}{(1-\gamma^2)}$	$\frac{2.9025E}{(1-\gamma^2)}$	$\frac{5.805E}{(1-\gamma^2)}$

becomes as the following expression:

$$\sigma_y = 0.0645 \frac{nE}{(1-\gamma^2)}. \quad (18)$$

3. CONCLUSIONS

The conclusions of this work may be summarized as follow.

- In the present guess, a novel expression of the yield strength of a bulk material is proposed on the basis of the indentation of a sharp needle of cone tip and using the Love (1939) and by Sneddon (1965) expression of the applied load.
- We have chosen an indenter of sharp needle form of cone tip of small the semi-included angle, $\theta \leq \pi/18$, to appreciate the hardness of tested material σ_y by using a neglect charge on the needle indenter and reduce the phenomena occurring during and after the tests, as cracking, deformation, *etc.*.
- The most important results of using the sharp needle-shaped indenter are the measuring of the elastic material strength and deriving its mathematical expression.
- The geometric and mathematical approximation of the indentation of needle of a sharp cone tip that presented in this work is revealed to establish the expression of the yield strength of solid material and the relationship provides a simple equation to evaluate directly the yield strength expression of an elastic solid from the Young's modulus E and the Poisson's ratio γ of the tested material.
- For a very pointed indenter of needle form of the cone tip ($R \ll H$), the yield strength of material can be expressed as the follow form:

$$\sigma_y = \frac{0.2026E}{(1-\gamma^2)\sin(\theta)}.$$

- The main conclusion is that application of a needle-shaped indenter put forward a new theoretical expression of the material hardness and experimental findings, which widen the scope of applications of various hardness test methods.
- Finally, we believe that the evaluation of the material hardness by the indentation of a sharp needle penetrator of cone-tip form is a very important theoretical and experimental study in area of the mechanical characterization of materials and the field of the contact mechanic, which will carry to extend the fields of application of the hardness tests.

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AUTHORS' CONTRIBUTIONS

The contributions of the first author are formulating ideas, hypotheses, the formulation and evolution of overarching research goals and aims, writing preparation, creation, and presentation of the published work. The second and the fourth authors reviewed the intellectual content and verified the methodology. As of the third and the last authors, their role is the verification of results, reproducibility, and the reference collection.

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